

A global attractor for a fluid–plate interaction model

Igor Chueshov* and Iryna Ryzhkova†

Department of Mechanics and Mathematics,
Kharkov National University,
Kharkov, 61077, Ukraine

September 21, 2011

Abstract

We study asymptotic dynamics of a coupled system consisting of linearized 3D Navier–Stokes equations in a bounded domain and a classical (nonlinear) elastic plate equation for transversal displacement on a flexible flat part of the boundary. We show that this problem generates a semiflow on appropriate phase space. Our main result states the existence of a compact finite-dimensional global attractor for this semiflow. We do not assume any kind of mechanical damping in the plate component. Thus our results means that dissipation of the energy in the fluid due to viscosity is sufficient to stabilize the system. To achieve the result we first study the corresponding linearized model and show that this linear model generates strongly continuous exponentially stable semigroup.

Keywords: Fluid–structure interaction, linearized 3D Navier–Stokes equations, nonlinear plate, finite-dimensional attractor.

2010 MSC: 74F10, 35B41, 35Q30, 74K20

1 Introduction

We consider a coupled (hybrid) system which describes interaction of a homogeneous viscous incompressible fluid which occupies a domain \mathcal{O} bounded by the (solid) walls of the container S and a horizontal boundary Ω on which a thin (nonlinear) elastic plate is placed. The motion of the fluid is described by linearized 3D Navier–Stokes equations. To describe deformations of the plate we consider a generalized plate model which accounts only for transversal displacements and covers a general large deflection Karman type model

*e-mail: chueshov@univer.kharkov.ua

†e-mail: iryonok@gmail.com

(see, e.g., [24, 25, 26] and also [15] and the references therein). However, our results can be also applied in the cases of nonlinear Berger and Kirchhoff plates (see the discussion in Section 4.1).

This fluid–structure interaction model assumes that large deflections of the plate produce small effect on the fluid. This corresponds to the case when the fluid fills the container which is large in comparison with the size of the plate.

We note that the mathematical studies of the problem of fluid–structure interaction in the case of viscous fluids and elastic plates/bodies have a long history. We refer to [9, 19, 20, 21, 22] and the references therein for the case of plates/membranes, to [16] in the case of moving elastic bodies, and to [1, 2, 3, 6, 7, 18] in the case of elastic bodies with the fixed interface; see also the literature cited in these references.

Our mathematical model is formulated as follows.

Let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded domain with a sufficiently smooth boundary $\partial\mathcal{O}$. We assume that $\partial\mathcal{O} = \overline{\Omega} \cup \overline{S}$, where $\Omega \cap S = \emptyset$ and

$$\Omega \subset \{x = (x_1; x_2; 0) : x' \equiv (x_1; x_2) \in \mathbb{R}^2\}$$

with the smooth contour $\Gamma = \partial\Omega$ and S is a surface which lies in the subspace $\mathbb{R}_-^3 = \{x_3 \leq 0\}$. The exterior normal on $\partial\mathcal{O}$ is denoted by n . We have that $n = (0; 0; 1)$ on Ω . We consider the following *linear* Navier–Stokes equations in \mathcal{O} for the fluid velocity field $v = v(x, t) = (v^1(x, t); v^2(x, t); v^3(x, t))$ and for the pressure $p(x, t)$:

$$v_t - \nu \Delta v + \nabla p = G_f(t) \quad \text{in } \mathcal{O} \times (0, +\infty), \quad (1)$$

$$\operatorname{div} v = 0 \quad \text{in } \mathcal{O} \times (0, +\infty), \quad (2)$$

where $\nu > 0$ is the dynamical viscosity and $G_f(t)$ is a volume force (which may depend on t). We supplement (1) and (2) with the (non-slip) boundary conditions imposed on the velocity field $v = v(x, t)$:

$$v = 0 \quad \text{on } S; \quad v \equiv (v^1; v^2; v^3) = (0; 0; u_t) \quad \text{on } \Omega. \quad (3)$$

Here $u = u(x, t)$ is the transversal displacement of the plate occupying Ω and satisfying the following equation (see, e.g., [8, 24, 25, 26] and the references therein):

$$u_{tt} + \Delta^2 u + \mathcal{F}(u) = G_{pl}(t) - T_f(v) \quad \text{in } \Omega \times (0, \infty),$$

where $G_{pl}(t)$ is a given body force on the plate, $\mathcal{F}(u)$ is a nonlinear feedback force which would be specified later and $T_f(v)$ is a surface force exerted by the fluid on the plate, $T_f(v) = (Tn|_{\Omega}, n)_{\mathbb{R}^3}$, where n is a outer unit normal to $\partial\mathcal{O}$ at Ω and $T = \{T_{ij}\}_{i,j=1}^3$ is the stress tensor of the fluid,

$$T_{ij} \equiv T_{ij}(v) = \nu \left(v_{x_j}^i + v_{x_i}^j \right) - p \delta_{ij}, \quad i, j = 1, 2, 3.$$

Since $n = (0; 0; 1)$ on Ω , we have that $T_f(v) = 2\nu\partial_{x_3}v^3 - p$. It also follows from (2) and (3) that $\partial_{x_3}v^3 = 0$ on Ω and thus we arrive at the equation

$$u_{tt} + \Delta^2 u + \mathcal{F}(u) = G_{pl}(t) + p|_{\Omega} \quad \text{in } \Omega \times (0, \infty). \quad (4)$$

We impose clamped boundary conditions on the plate

$$u|_{\partial\Omega} = \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0 \quad (5)$$

and supply (1)–(5) with initial data of the form

$$v(0) = v_0, \quad u(0) = u_0, \quad u_t(0) = u_1, \quad (6)$$

We note that (2) and (3) imply the following compatibility condition

$$\int_{\Omega} u_t(x', t) dx' = 0 \quad \text{for all } t \geq 0. \quad (7)$$

This condition fulfills when

$$\int_{\Omega} u(x', t) dx' = \text{const} \quad \text{for all } t \geq 0,$$

which can be interpreted as preservation of the volume of the fluid.

We also note that a similar class of models was considered before in [11, 19, 20, 21]. The main difference between (1)–(6) and models in these publications is that the papers mentioned deal *only* with longitudinal deformations of the plate neglecting transversal deformations (in contrast with the model (1)–(6) which takes into account the transversal deformations only). This means that instead of (3) the following boundary conditions are imposed on the velocity fluid field:

$$v = 0 \quad \text{on } S; \quad v \equiv (v^1; v^2; v^3) = (u_t^1; u_t^2; 0) \quad \text{on } \Omega, \quad (8)$$

where $u = (u^1(x, t); u^2(x, t))$ is the in-plane displacement vector of the plate which solves the wave equation of the form

$$u_{tt} - \Delta u - \nabla [\text{div } u] + \nu(v_{x_3}^1; v_{x_3}^2)|_{x_3=0} + f(u) = 0 \quad \text{in } \Omega; \quad u^i = 0 \quad \text{on } \Gamma. \quad (9)$$

This kind of models arises in the study of blood flows in large arteries (see the references in [19]). The model (1), (2), (8), (9) is simpler in several respects. One of them is related to the fact the force exerted on the plate by the fluid is more regular in the case (9) and does not contains the pressure in an explicit form. Moreover, the model (1), (2), (8), (9) does not require any compatibility conditions like (7), because the volume of the fluid obviously preserves in the case of longitudinal deformations.

In this paper our main point of interest is well-posedness and long-time dynamics of solutions to the coupled problem in (1)–(6) for the velocity v and the displacement u . First we consider the linear version of this problem (i.e., the case when $\mathcal{F}(u) \equiv 0$). For this linear version we prove well-posedness in the class of weak (energy) solutions and establish some additional properties of solutions which we need for treating the nonlinear problem. In particular, we show that in the homogeneous case ($G_f \equiv 0$, $G_{pl} \equiv 0$) the linear version generates strongly continuous exponentially stable semigroup. Then we consider a nonlinear version of this problem under rather general hypotheses concerning nonlinearity. These hypotheses cover the cases of von Karman, Berger and Kirchhoff plates. We show that problem (1)–(6) generates a dynamical system in an energy type space. Our main result (see Theorem 4.8) states that under some natural conditions concerning feedback forces system (1)–(6) possesses a compact global attractor of finite fractal dimension. To establish this results we rely on recently developed approach (see [13], [14] and [15, Chapters 7,8] and also the references therein) which involves stabilizability estimates and notion of a quasi-stable system.

The paper is organized as follows. In Section 2 we introduce notations, recall some properties of Sobolev type spaces with non-integer indexes on bounded domains and collect some regularity properties of (stationary) Stokes problem which we use in the further considerations (see Proposition 2.2). Section 3 is devoted to a linear version of the problem. Our main result in this section is Theorem 3.3 on well-posedness of weak solutions. In Section 4 we deal with the nonlinear problem (1)–(6). First we prove well-posedness result in Theorem 4.3 and then show that in the case of autonomous forces the problem generates a gradient dynamical system. Our main result in this section states existence of a finite dimensional global attractor and describes some regularity properties of the trajectories from the attractor. The argument is based on the quasi-stability property established in Proposition 4.10.

2 Preliminaries

In this section we introduce Sobolev type spaces we need and provide with some results concerning to Stokes problem.

2.1 Spaces and notations

To introduce Sobolev spaces we follow approach presented in [33].

Let D be a sufficiently smooth domain and $s \in \mathbb{R}$. We denote by $H^s(D)$ the Sobolev space of order s on a set D which we define as restriction (in the sense of distributions) of the space $H^s(\mathbb{R}^d)$ (introduced via Fourier transform). We denote by $\|\cdot\|_{s,D}$ the norm in $H^s(D)$ which we define by

the relation

$$\|u\|_{s,D}^2 = \inf \left\{ \|w\|_{s,\mathbb{R}^d}^2 : w \in H^s(\mathbb{R}^d), w = u \text{ on } D \right\}$$

We also use the notation $\|\cdot\|_D = \|\cdot\|_{0,D}$ for the corresponding L_2 -norm and, similarly, $(\cdot, \cdot)_D$ for the L_2 inner product. We denote by $H_0^s(D)$ the closure of $C_0^\infty(D)$ in $H^s(D)$ (with respect to $\|\cdot\|_{s,D}$) and introduce the spaces

$$H_*^s(D) := \left\{ f|_D : f \in H^s(\mathbb{R}^d), \text{ supp } f \subset \overline{D} \right\}, \quad s \in \mathbb{R}.$$

Since the extension by zero of elements from $H_*^s(D)$ gives us an element of $H^s(\mathbb{R}^d)$, these spaces $H_*^s(D)$ can be treated not only as functional spaces defined on D (and contained in $H^s(D)$) but also as (closed) subspaces of $H^s(\mathbb{R}^d)$. Below we need them to describe boundary traces on $\Omega \subset \partial\mathcal{O}$. We endow the classes $H_*^s(D)$ with the induced norms $\|f\|_{s,D}^* = \|f\|_{s,\mathbb{R}^d}$ for $f \in H_*^s(D)$. It is clear that

$$\|f\|_{s,D} \leq \|f\|_{s,D}^*, \quad f \in H_*^s(D).$$

It is known (see [33, Theorem 4.3.2/1]) that $C_0^\infty(D)$ is dense in $H_*^s(D)$ and

$$\begin{aligned} H_*^s(D) &\subset H_0^s(D) \subset H^s(D), \quad s \in \mathbb{R}; \\ H_0^s(D) &= H^s(D), \quad -\infty < s \leq 1/2; \\ H_*^s(D) &= H_0^s(D), \quad -1/2 < s < \infty, \quad s - 1/2 \notin \{0, 1, 2, \dots\}. \end{aligned}$$

In particular, $H_*^s(D) = H_0^s(D) = H^s(D)$ for $|s| < 1/2$. By [33, Remark 4.3.2/2] we also have that $H_*^s(D) \neq H^s(D)$ for $|s| > 1/2$. Note that in the notations of [27] the space $H_*^{m+1/2}(D)$ is the same as $H_{00}^{m+1/2}(D)$ for every $m = 0, 1, 2, \dots$, and for $s = m + \sigma$ with $0 < \sigma < 1$ we have

$$\|u\|_{s,D}^* = \left\{ \|u\|_{s,D}^2 + \sum_{|\alpha|=m} \int_D \frac{|D^\alpha u(x)|^2}{d(x, \partial D)^{2\sigma}} dx \right\}^{1/2},$$

where $d(x, \partial D)$ is the distance between x and ∂D . The norm $\|\cdot\|_{s,D}^*$ is equivalent to $\|\cdot\|_{s,D}$ in the case when $s > -1/2$ and $s - 1/2 \notin \{0, 1, 2, \dots\}$, but not equivalent in general.

Understanding adjoint spaces with respect to duality between $C_0^\infty(D)$ and $[C_0^\infty(D)]'$ by Theorems 4.8.1 and 4.8.2 from [33] we also have that

$$[H_*^s(D)]' = H^{-s}(D), \quad s \in \mathbb{R}, \quad \text{and} \quad [H^s(D)]' = H_*^{-s}(D), \quad s \in (-\infty, 1/2).$$

Below we also use the factor-spaces $H^s(D)/\mathbb{R}$ with the naturally induced norm.

To describe fluid velocity fields we introduce the following spaces.

Let $\mathcal{C}(\mathcal{O})$ be the class of C^∞ vector-valued solenoidal (i.e., divergence-free) functions $v = (v^1; v^2; v^3)$ on $\overline{\mathcal{O}}$ which vanish in a neighborhood of S and such that $v^1 = v^2 = 0$ on Ω . We denote by X the closure of $\mathcal{C}(\mathcal{O})$ with respect to the L_2 -norm and by V the closure with respect to the $H^1(\mathcal{O})$ -norm. One can see that

$$X = \{v = (v^1; v^2; v^3) \in [L_2(\mathcal{O})]^3 : \operatorname{div} v = 0; \gamma_n v \equiv (v, n) = 0 \text{ on } S\}$$

and

$$V = \left\{ v = (v^1; v^2; v^3) \in [H^1(\mathcal{O})]^3 \mid \begin{array}{l} \operatorname{div} v = 0, \ v = 0 \text{ on } S, \\ v^1 = v^2 = 0 \text{ on } \Omega \end{array} \right\}.$$

We equip X with L_2 -type norm $\|\cdot\|_{\mathcal{O}}$ and denote by $(\cdot, \cdot)_{\mathcal{O}}$ the corresponding inner product. The space V is endowed with the norm $\|\cdot\|_V = \|\nabla \cdot\|_{\mathcal{O}}$. For some details concerning this type spaces we refer to [32], for instance.

We also need the Sobolev spaces consisting of functions with zero average on the domain Ω , namely we consider the space

$$\widehat{L}_2(\Omega) = \left\{ u \in L_2(\Omega) : \int_{\Omega} u(x') dx' = 0 \right\}$$

and also $\widehat{H}^s(\Omega) = H^s(\Omega) \cap \widehat{L}_2(\Omega)$ for $s > 0$ with the standard $H^s(\Omega)$ -norm. The notations $\widehat{H}_*^s(\Omega)$ and $\widehat{H}_0^s(\Omega)$ have a similar meaning.

Remark 2.1 Below we use $\widehat{H}_0^2(\Omega)$ as a state space for the displacement of the plate. It is clear that $\widehat{H}_0^2(\Omega)$ is a closed subspace of $H_0^2(\Omega)$. We denote by \widehat{P} the projection on $\widehat{H}_0^2(\Omega)$ in $H_0^2(\Omega)$ which is orthogonal with respect to the inner product $(\Delta \cdot, \Delta \cdot)_{\Omega}$. One can see that $(I - \widehat{P})H_0^2(\Omega)$ consists of functions $u \in H_0^2(\Omega)$ such that $\Delta^2 u = \text{const}$ and thus has dimension one.

2.2 Stokes problem

In further considerations we need some regularity properties of the terms responsible for fluid–plate interaction. To this end we consider the following Stokes problem

$$\begin{aligned} -\nu \Delta v + \nabla p &= g, \quad \operatorname{div} v = 0 \quad \text{in } \mathcal{O}; \\ v &= 0 \quad \text{on } S; \quad v = (0; 0; \psi) \quad \text{on } \Omega, \end{aligned} \tag{10}$$

where $g \in [L^2(\mathcal{O})]^3$ and $\psi \in \widehat{L}_2(\Omega)$ are given. This type of boundary value problems for the Stokes equation was studied by many authors (see, e.g., [23] and [32] and the references therein). We collect some properties of solutions to (10) in the following assertion.

Proposition 2.2 *With the reference to problem (10) the following statements hold.*

- (1) Let $g \in [H^{-1+\sigma}(\mathcal{O})]^3$ and $\psi \in H_*^{1/2+\sigma}(\Omega)$ be such that $\int_{\Omega} \psi(x') dx' = 0$. Then for every $0 \leq \sigma \leq 1$ problem (10) has a unique solution $\{v; p\}$ in $[H^{1+\sigma}(\mathcal{O})]^3 \times [H^{\sigma}(\mathcal{O})/\mathbb{R}]$ such that

$$\|v\|_{[H^{1+\sigma}(\mathcal{O})]^3} + \|p\|_{H^{\sigma}(\mathcal{O})/\mathbb{R}} \leq c_0 \left\{ \|g\|_{[H^{-1+\sigma}(\mathcal{O})]^3} + \|\psi\|_{H_*^{\sigma+1/2}(\Omega)} \right\}. \quad (11)$$

- (2) If $g = 0$, $\psi \in H_*^{-1/2+\sigma}(\Omega)$, $0 \leq \sigma \leq 1$, $\int_{\Omega} \psi dx' = 0$, then

$$\|v\|_{[H^{\sigma}(\mathcal{O})]^3} + \|p\|_{H^{-1+\sigma}(\mathcal{O})/\mathbb{R}} \leq c_0 \|\psi\|_{H_*^{-1/2+\sigma}(\Omega)}. \quad (12)$$

In particular, we can define a linear operator $N_0 : \widehat{L}_2(\Omega) \mapsto [H^{1/2}(\mathcal{O})]^3$ by the formula

$$N_0 \psi = w \quad \text{iff} \quad \begin{cases} -\nu \Delta w + \nabla p = 0, & \operatorname{div} w = 0 \quad \text{in } \mathcal{O}; \\ w = 0 \quad \text{on } S; & w = (0; 0; \psi) \quad \text{on } \Omega, \end{cases} \quad (13)$$

for $\psi \in \widehat{L}_2(\Omega)$ ($N_0 \psi$ solves (10) with $g \equiv 0$). It follows from (11) and (12) that

$$N_0 : \widehat{H}_*^s(\Omega) \mapsto [H^{1/2+s}(\mathcal{O})]^3 \cap X \quad \text{continuously for } -\frac{1}{2} \leq s \leq \frac{3}{2}.$$

- (3) Let $g \in [H^{-1/2+\sigma}(\mathcal{O})]^3$ and $\psi \in \widehat{H}_*^{\sigma}(\Omega)$, with $0 < \sigma \leq 1/2$. Then we can define the trace of the pressure p on Ω , which possesses the property $p|_{\Omega} \in H^{-1+\sigma}(\Omega)/\mathbb{R}$ and

$$\|p\|_{H^{-1+\sigma}(\Omega)/\mathbb{R}} \leq c_0 \left\{ \|g\|_{[H^{-1/2+\sigma}(\mathcal{O})]^3} + \|\psi\|_{\widehat{H}_*^{\sigma}(\Omega)} \right\}. \quad (14)$$

Proof. Since the extension of elements from $H_*^{\sigma}(\Omega)$ by zero to the whole boundary $\partial\mathcal{O}$ do not change the smoothness Sobolev class, i.e., leads to elements from $H^s(\partial\mathcal{O})$, we can use the regularity results available for the Stokes problem with the Dirichlet type boundary conditions imposed on the whole $\partial\mathcal{O}$ (see, e.g., [23, 32] and also the paper [17] and the references therein). This observation leads to the following arguments.

1. The existence and uniqueness of solutions along with the bound in (11) follow from Proposition 2.3 and Remark 2.6 on Sobolev norm's interpolation in [32, Chapter 1].

2. By Theorem 3[17] (applied for the boundary data $\tilde{\psi} \in \widehat{H}^{-1/2}(\partial\mathcal{O})$ which is extension by zero outside Ω of the function $\psi \in \widehat{H}_*^{-1/2}(\Omega)$) we have (12) with $\sigma = 0$. Therefore interpolating with (11) for $s = 0$ with $g \equiv 0$ we obtain (12) for all $0 \leq \sigma \leq 1$.

3. We first represent v in the form $v = \hat{v} + v^*$, where \hat{v} solves (10) with $\psi \equiv 0$ and v^* satisfies (10) with $g \equiv 0$. Let \hat{p} and p^* be the corresponding

representatives of the pressure (which are identified with an element in a factor-space). By the first statement we have that $\hat{p} \in H^{1/2+\sigma}(\mathcal{O})$ and thus by the standard trace theorem there exists $\hat{p}|_{\partial\mathcal{O}} \in H^\sigma(\partial\mathcal{O})$. This implies that $\hat{p}|_\Omega \in H^\sigma(\Omega) \subset H^{-1+\sigma}(\Omega)$ and

$$\|\hat{p}\|_{H^{-1+\sigma}(\Omega)/\mathbb{R}} \leq c\|\hat{p}\|_{H^\sigma(\Omega)/\mathbb{R}} \leq c\|g\|_{[H^{-1/2+\sigma}(\mathcal{O})]^3}. \quad (15)$$

In the case $g \equiv 0$ the pressure p^* is a harmonic function in \mathcal{O} which belongs $H^{-1/2+\sigma}(\mathcal{O})$. This allows us to assign a meaning to $p^*|_\Omega$ in $H^{-1+\sigma}(\Omega)$. Indeed, let $\phi \in C_0^\infty(\Omega)$ and $\tilde{\phi} \in C_0^\infty(\partial\mathcal{O})$ be the extension of ϕ by zero. Then by the trace theorem there exists a smooth function w_ϕ on \mathcal{O} such that

$$w_\phi|_{\partial\mathcal{O}} = 0, \quad \frac{\partial w_\phi}{\partial n}\Big|_{\partial\mathcal{O}} = \tilde{\phi}, \quad \|w_\phi\|_{H^{5/2-\sigma}(\mathcal{O})} \leq C\|\phi\|_{H_*^{1-\sigma}(\Omega)}.$$

The application of Green's formula yields $(p^*, \Delta w_\phi)_\mathcal{O} = (p^*, \phi)_\Omega$. Therefore

$$|(\phi, p^*)_\Omega| = |(p^*, \Delta w_\phi)_\mathcal{O}| \leq C\|p^*\|_{-1/2+\sigma, \mathcal{O}}\|\tilde{\phi}\|_{1-\sigma, \partial\mathcal{O}}.$$

Since $\|\tilde{\phi}\|_{1-\sigma, \partial\mathcal{O}} = \|\phi\|_{H_*^{1-\sigma}(\Omega)}$ and $C_0^\infty(\Omega)$ is dense in $H_*^{1-\sigma}(\Omega)$, we obtain

$$\|p^*\|_{H^{-1+\sigma}(\Omega)/\mathbb{R}} \leq c\|p^*\|_{H^{-1/2+\sigma}(\mathcal{O})/\mathbb{R}} \leq c\|\psi\|_{H_*^\sigma(\Omega)}. \quad (16)$$

Thus relation (14) follows from (15) and (16). \square

3 Linear problem

In this section we consider a linear version of (1)–(6) which is obtained from (1)–(6) by replacing equation (4) with its linear version. Thus we deal with the following problem

$$v_t - \nu\Delta v + \nabla p = G_f(t) \quad \text{and} \quad \operatorname{div} v = 0 \quad \text{in} \quad \mathcal{O} \times (0, +\infty), \quad (17)$$

$$v = 0 \quad \text{on} \quad S \quad \text{and} \quad v \equiv (v^1; v^2; v^3) = (0; 0; u_t) \quad \text{on} \quad \Omega, \quad (18)$$

$$u_{tt} + \Delta^2 u = G_{pl}(t) + p|_\Omega \quad \text{on} \quad \Omega, \quad (19)$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega, \quad (20)$$

which we supply with the initial data of the form

$$v(0) = v_0, \quad u(0) = u_0, \quad u_t(0) = u_1. \quad (21)$$

To define weak (variational) solutions we need the following class \mathcal{L}_T of test functions ϕ on \mathcal{O} :

$$\mathcal{L}_T = \left\{ \phi \left| \begin{array}{l} \phi \in L_2(0, T; [H^1(\mathcal{O})]^3), \quad \phi_t \in L_2(0, T; [L_2(\mathcal{O})]^3), \\ \operatorname{div} \phi = 0, \quad \phi|_S = 0, \quad \phi|_\Omega = (0; 0; b), \quad \phi(T) = 0, \\ b \in L_2(0, T; \widehat{H}_0^2(\Omega)), \quad b_t \in L_2(0, T; \widehat{L}_2(\Omega)). \end{array} \right. \right\}$$

Definition 3.1 A pair of functions $(v(t); u(t))$ is said to be a weak solution to the problem in (17)–(21) on a time interval $[0, T]$ if

- $v \in L_\infty(0, T; X) \cap L_2(0, T; V)$;
- $u \in L_\infty(0, T; H_0^2(\Omega))$, $u_t \in L_\infty(0, T; \widehat{L}_2(\Omega))$ and $u(0) = u_0$;
- for every $\phi \in \mathcal{L}_T$ the following equality holds:

$$\begin{aligned} & - \int_0^T (v, \phi_t)_\mathcal{O} dt + \nu \int_0^T (\nabla v, \nabla \phi)_\mathcal{O} dt - \int_0^T (u_t, b_t)_\Omega dt + \int_0^T (\Delta u, \Delta b)_\Omega dt \\ & = \int_0^T (G_f(t), \phi)_\mathcal{O} dt + \int_0^T (G_{pl}(t), b)_\Omega dt + (v_0, \phi(0))_\mathcal{O} + (u_1, b(0))_\Omega; \end{aligned} \quad (22)$$

- the compatibility condition $v(t)|_\Omega = (0; 0; u_t(t))$ holds for almost all t .

Remark 3.2 (1) It follows from the compatibility condition and the standard trace theorem that $u_t \in L_2(0, T; H_*^{1/2}(\Omega))$ and

$$\|u_t(t)\|_{H_*^{1/2}(\Omega)} \leq C \|\nabla v(t)\|_\mathcal{O} \quad \text{for almost all } t \in [0, T].$$

(2) Taking in (22) $\phi(t) = \int_t^T \chi(\tau) d\tau \cdot \psi$, where χ is a smooth scalar function and ψ belongs to the space

$$W = \left\{ \psi \in V \mid \psi|_\Omega = (0; 0; \beta), \beta \in \widehat{H}_0^2(\Omega) \right\}, \quad (23)$$

one see that the weak solution $(v(t); u(t))$ satisfies the relation

$$\begin{aligned} (v(t), \psi)_\mathcal{O} + (u_t(t), \beta)_\Omega &= (v_0, \psi)_\mathcal{O} + (u_1, \beta)_\Omega \\ &- \int_0^t [\nu(\nabla v, \nabla \psi)_\mathcal{O} + (\Delta u, \Delta \beta)_\Omega - (G_f, \psi)_\mathcal{O} - (G_{pl}, \beta)_\Omega] d\tau \end{aligned} \quad (24)$$

for almost all $t \in [0, T]$ and for all $\psi = (\psi^1; \psi^2; \psi^3) \in W$, where $\beta = \psi^3|_\Omega$.

Below as a phase space we use

$$\mathcal{H} = \left\{ (v_0; u_0; u_1) \in X \times H_0^2(\Omega) \times \widehat{L}_2(\Omega) : (v_0, n) \equiv v_0^3 = u_1 \text{ on } \Omega \right\} \quad (25)$$

with the norm $\|(u_0; u_0; u_1)\|_\mathcal{H}^2 = \|v_0\|_\mathcal{O}^2 + \|\Delta u_0\|_\Omega^2 + \|u_1\|_\Omega^2$. We also denote by $\widehat{\mathcal{H}}$ a subspace in \mathcal{H} of the form

$$\widehat{\mathcal{H}} = \left\{ (v_0; u_0; u_1) \in \mathcal{H} : u_0 \in \widehat{H}_0^2(\Omega) \right\}. \quad (26)$$

Our main result in this section is the following well-posedness theorem concerning the linear problem.

Theorem 3.3 Assume that $U_0 = (v_0; u_0; u_1) \in \mathcal{H}$, $G_f(t) \in L_2(0, T; V')$ and $G_{pl}(t) \in L_2(0, T; H^{-1/2}(\Omega))$. Then for any interval $[0, T]$ there exists a unique weak solution $(v(t); u(t))$ to (17)–(21) with the initial data U_0 . This solution possesses the property

$$U(t; U_0) \equiv U(t) \equiv (v(t); u(t); u_t(t)) \in C(0, T; X \times H_0^2(\Omega) \times \widehat{L}_2(\Omega)), \quad (27)$$

and satisfies the energy balance equality

$$\begin{aligned} \mathcal{E}_0(v(t), u(t), u_t(t)) + \nu \int_0^t \|\nabla v\|_{\mathcal{O}}^2 d\tau &= \mathcal{E}_0(v_0, u_0, u_1) \\ &+ \int_0^t (G_f(\tau), v)_{\mathcal{O}} d\tau + \int_0^t (G_{pl}(\tau), u_\tau)_\Omega d\tau \end{aligned} \quad (28)$$

for every $t > 0$, where the energy functional \mathcal{E}_0 is defined by the relation

$$\mathcal{E}_0(v(t), u(t), u_t(t)) = \frac{1}{2} (\|v(t)\|_{\mathcal{O}}^2 + \|u_t(t)\|_\Omega^2 + \|\Delta u(t)\|_\Omega^2). \quad (29)$$

Moreover, there exist positive constants M and γ such that for every initial data $U_0 = (v_0; u_0; u_1)$ from $\widehat{\mathcal{H}}$ we have

$$\|U(t)\|_{\mathcal{H}}^2 \leq M e^{-\gamma t} \|U_0\|_{\mathcal{H}}^2 + M \int_0^t e^{-\gamma(t-\tau)} [\|G_f(\tau)\|_{V'}^2 + \|G_{pl}(\tau)\|_{-1/2, \Omega}^2] d\tau \quad (30)$$

Remark 3.4 Let $w_0 \in (I - \widehat{P})H_0^2(\Omega)$, where the projector \widehat{P} is defined in Remark 2.1. Then one can see that the pair $\{v(t) \equiv 0, u(t) \equiv w_0\}$ solve problem (17)–(21) with the initial data $(0; w_0; 0)$ and with $G_f \equiv 0$, $G_{pl} \equiv 0$. The pressure p is the constant determined from its boundary value on Ω : $p|_\Omega = \Delta^2 w_0$ ($\Delta^2 w_0$ is a constant due to Remark 2.1). This observation gives us a relation between solutions with initial data from \mathcal{H} and $\widehat{\mathcal{H}}$, namely we have that

$$U(t; (v_0; u_0; u_1)) = U(t; (v_0; \widehat{P}u_0; u_1)) + (0; (I - \widehat{P})u_0; 0), \quad t > 0,$$

for any $U_0 = (v_0; u_0; u_1) \in \mathcal{H}$. This relation means that $\widehat{\mathcal{H}}$ is invariant with respect to dynamics governed by (17)–(21) and explains why an exponential decay estimate of the form (30) cannot be true for *every* initial data $U_0 = (v_0; u_0; u_1)$ from the space \mathcal{H} .

This remark allows us to derive from Theorem 3.3 the following assertion.

Corollary 3.5 Problem (17)–(21) with $G_f \equiv 0$ and $G_{pl} \equiv 0$ generates a strongly continuous contraction semigroup T_t on \mathcal{H} and on $\widehat{\mathcal{H}}$ by the formula $T_t U_0 = U(t)$, where $U(t)$ is a weak solution to (17)–(21) with the initial data U_0 . This semigroup T_t is exponentially stable on $\widehat{\mathcal{H}}$, i.e., there exist positive constants M and γ such that

$$\|T_t U_0\|_{\mathcal{H}} \leq M e^{-\gamma t} \|U_0\|_{\mathcal{H}} \quad \text{for any } U_0 = (v_0; u_0; u_1) \in \widehat{\mathcal{H}}.$$

Proof. Strong continuity of T_t follows from (27). This semigroup is contractive and exponentially stable due to (28) and (30) with $G_f \equiv 0$ and $G_{pl} \equiv 0$. \square

We note that the generator of the semigroup T_t defined via solutions to problem (17)–(21) in the space $\hat{\mathcal{H}}$ has a rather complicated structure, see Appendix A in the end of the paper. This is why we avoid in the argument below calculations involving the explicit form of the generator.

Proof of Theorem 3.3

We use the compactness method and split the argument into several steps.

Step 1. Existence of an approximate solution. For the construction of Galerkin's approximations we use an idea of [9] in a slightly modified form.

Let $\{\psi_i\}_{i \in \mathbb{N}}$ be the orthonormal basis in $\tilde{X} = \{v \in X : (v, n)|_{\Omega} = 0\}$ consisting of the eigenvectors of the Stokes problem:

$$-\Delta \psi_i + \nabla p_i = \mu_i \psi_i \quad \text{in } \mathcal{O}, \quad \operatorname{div} \psi_i = 0, \quad \psi_i|_{\partial \mathcal{O}} = 0,$$

where $0 < \mu_1 \leq \mu_2 \leq \dots$ are the corresponding eigenvalues. Denote by $\{\xi_i\}_{i \in \mathbb{N}}$ the basis in $\hat{H}_0^2(\Omega)$ which consists of eigenfunctions of the following problem

$$(\Delta \xi_i, \Delta w)_{\Omega} = \kappa_i (\xi_i, w)_{\Omega}, \quad \forall w \in \hat{H}_0^2(\Omega),$$

with the eigenvalues $0 < \kappa_1 \leq \kappa_2 \leq \dots$ and $\|\xi_i\|_{\Omega} = 1$. Let $\phi_i = N_0 \xi_i$, where the operator N_0 is defined by (13). By Proposition 2.2 $\phi_i \in [H^2(\mathcal{O})]^3 \cap V$. As above one can also see that $\partial_{x_3} \phi_i^3 = 0$ on Ω .

We define an approximate solution as a pair of functions

$$v_{n,m}(t) = \sum_{i=1}^m \alpha_i(t) \psi_i + \sum_{j=1}^n \dot{\beta}_j(t) \phi_j, \quad u_n(t) = \sum_{j=1}^n \beta_j(t) \xi_j + (I - \hat{P})u_0, \quad (31)$$

satisfying the relations

$$\dot{\alpha}_k(t) + \sum_{j=1}^n \ddot{\beta}_j(t) (\phi_j, \psi_k)_{\mathcal{O}} + \nu \mu_k \alpha_k(t) + \nu \sum_{j=1}^n \dot{\beta}_j(t) (\nabla \phi_j, \nabla \psi_k)_{\mathcal{O}} = (G_f, \psi_k)_{\mathcal{O}} \quad (32)$$

for $k = 1 \dots m$, and

$$\begin{aligned} & \sum_{i=1}^m \dot{\alpha}_i(t) (\psi_i, \phi_k)_{\mathcal{O}} + \sum_{j=1}^n \ddot{\beta}_j(t) (\phi_j, \phi_k)_{\mathcal{O}} + \ddot{\beta}_k(t) \\ & + \nu \sum_{i=1}^m \alpha_i(t) (\nabla \psi_i, \nabla \phi_k)_{\mathcal{O}} + \nu \sum_{j=1}^n \dot{\beta}_j(t) (\nabla \phi_j, \nabla \phi_k)_{\mathcal{O}} + \kappa_k \beta_k(t) \\ & = (G_f(t), \phi_k)_{\mathcal{O}} + (G_{pl}(t), \xi_k)_{\Omega} \quad (33) \end{aligned}$$

for $k = 1, \dots, n$. This system of ordinary differential equations is endowed with the initial data

$$\begin{aligned} v_{v,m}(0) &= \Pi_m(v_0 - N_0 u_1) + N_0 P_n u_1, \\ u_n(0) &= P_n \hat{P} u_0 + (I - \hat{P}) u_0, \quad \dot{u}_n(0) = P_n u_1, \end{aligned}$$

where Π_m is the orthoprojector on $Lin\{\psi_j : j = 1, \dots, m, \}$ in \tilde{X} and P_n is orthoprojector on $Lin\{\xi_i : i = 1, \dots, n\}$ in $\hat{L}_2(\Omega)$. Since Π_m and P_n are spectral projectors we have that

$$(v_{v,m}(0); u_n(0); \dot{u}_n(0)) \rightarrow (v_0; u_0; u_1) \text{ strongly in } \mathcal{H} \text{ as } m, n \rightarrow \infty. \quad (34)$$

We can rewrite system (32) and (33) as

$$M \frac{d}{dt} \begin{pmatrix} \alpha(t) \\ \dot{\beta}(t) \end{pmatrix} + g(\alpha(t), \beta(t), \dot{\beta}(t)) + G(t) = 0$$

for some linear function $g : \mathbb{R}^{m+2n} \mapsto \mathbb{R}^{m+n}$ and $G \in L_2(0, T; \mathbb{R}^{m+n})$, where

$$M = \begin{bmatrix} 0 & 0 \\ 0 & id \end{bmatrix} + \begin{bmatrix} \{(\psi_i, \psi_j)\}_{j,k=1}^m & \{(\psi_l, \phi_k)\}_{l,k=1}^{m,n} \\ \{(\phi_k, \psi_l)\}_{l,k=1}^{n,m} & \{(\phi_i, \phi_j)\}_{j,k=1}^n \end{bmatrix}. \quad (35)$$

The first matrix in (35) is nonnegative and the second one is symmetric and strictly positive (since the functions $\{\psi_i, \phi_j : i = 1, \dots, m, j = 1, \dots, n\}$ are linearly independent). Therefore system (32) and (33) has a unique solution on any time interval $[0, T]$.

It follows from (31) that

$$v_{n,m}(t) = \sum_{i=1}^m \alpha_i(t) \psi_i + N_0 [\partial_t u_n(t)],$$

where N_0 is given by (13). This implies the following boundary compatibility condition

$$v_{n,m}(t) = (0; 0; \partial_t u_n(t)) \text{ on } \Omega. \quad (36)$$

Step 2. Energy relation and a priori estimate for an approximate solution. It follows from (32) and (33) that the approximate solutions satisfy the relation

$$\begin{aligned} (\dot{v}_{n,m}(t), \chi)_{\mathcal{O}} + (\ddot{u}_n(t), h)_{\Omega} + \nu(\nabla v_{n,m}(t), \nabla \chi)_{\mathcal{O}} + (\Delta u_n(t), \Delta h)_{\Omega} \\ = (G_f(t), \chi)_{\mathcal{O}} + (G_{pl}(t), h)_{\Omega} \end{aligned} \quad (37)$$

for $t \in [0, T]$ and for every χ and h of the form

$$\chi(t) = \sum_{k=1}^{m'} \chi_k \psi_k + N_0 h \quad \text{with} \quad h = \sum_{k=1}^{n'} h_k \xi_k,$$

where $m' \leq m$ and $n' \leq n$. Therefore taking $\chi = v_{n,m}$ we obtain the following energy balance relation for approximate solutions

$$\begin{aligned} & \mathcal{E}_0(v_{n,m}(t), u_n(t), \partial_t u_n(t)) + \nu \int_0^t \int_{\mathcal{O}} |\nabla v_{n,m}|^2 dx d\tau \\ &= \mathcal{E}_0(v_{n,m}(0), u_n(0), \partial_t u_n(0)) + \int_0^t (G_f, v_{n,m})_{\mathcal{O}} d\tau + \int_0^t (G_{pl}, \partial_t u_n)_{\Omega} d\tau. \end{aligned} \quad (38)$$

This implies the following a priori estimate

$$\sup_{t \in [0, T]} \{ \|v_{n,m}(t)\|_{\mathcal{O}}^2 + \|\Delta u_n(t)\|_{\Omega}^2 + \|\partial_t u_n(t)\|_{\Omega}^2 \} + \int_0^T \|\nabla v_{n,m}\|_{\mathcal{O}}^2 d\tau \leq C_T. \quad (39)$$

By the trace theorem from (36) we also have that

$$\int_0^T \|\partial_t u_n(\tau)\|_{H_*^{1/2}(\Omega)}^2 d\tau = \int_0^T \|v_{n,m}(\tau)\|_{1/2, \partial\mathcal{O}}^2 d\tau \leq C_T. \quad (40)$$

Step 3. Limit transition. By (39) the sequence $\{(v_{n,m}; u_n; \partial_t u_n)\}$ contains a subsequence such that

$$(v_{n,m}; u_n; \partial_t u_n) \rightharpoonup (v; u; \partial_t u) \quad * \text{-weakly in } L_{\infty}(0, T; \mathcal{H}); \quad (41)$$

$$u_n \rightarrow u \quad \text{strongly in } C(0, T; H_0^{2-\epsilon}(\Omega)), \quad \forall \epsilon > 0; \quad (42)$$

$$v_{n,m} \rightharpoonup v \quad \text{weakly in } L_2(0, T; V). \quad (43)$$

To obtain (42) we use the Aubin-Dubinsky theorem (see, e.g., [30, Corollary 4]). By (40) we can also suppose that

$$\partial_t u_n \rightharpoonup \partial_t u \quad \text{weakly in } L_2(0, T; H_*^{1/2}(\Omega)); \quad (44)$$

$$v_{n,m} \rightharpoonup v \quad \text{weakly in } L_2(0, T; H^{1/2}(\partial\mathcal{O})). \quad (45)$$

One can see from (37) that $(v_{n,m}; u_n; \partial_t u_n)(t)$ satisfies (22) with the test function ϕ of the form

$$\phi = \phi_{p,q} = \sum_{i=1}^p \gamma_i(t) \psi_i + \sum_{j=1}^q \delta_j(t) \phi_j, \quad (46)$$

where $p \leq m$, $q \leq n$ and γ_i, δ_j are scalar absolutely continuous functions on $[0, T]$ such that $\dot{\gamma}_i, \dot{\delta}_j \in L_2(0, T)$ and $\gamma_i(T) = \delta_j(T) = 0$. Thus using (41)–(43) we can pass to the limit and show that $(v; u; \partial_t u)(t)$ satisfies (22) with $\phi = \phi_{p,q}$, where p and q are arbitrary. By (34) and (42) we have $u(0) = u_0$. The compatibility condition (18) follows from (36) and (44), (45).

To conclude the proof of the existence of weak solutions we only need to show that any function ϕ in \mathcal{L}_T can be approximate by a sequence of functions of the form (46). This can be done in the following way. We

first approximate the corresponding boundary value of b by a finite linear combination h of ξ_j , then we approximate the difference $\phi - N_0 h$ (with N_0 define by (13)) by finite linear combination of ψ_k .

Thus the existence of weak solutions is proved. One can also see from (38) and from (41)–(43) that the constructed weak solution satisfies the corresponding energy balance *inequality*.

Step 4. Uniqueness. We use the same idea as in [28], but with a slightly modified test function, see (48).

Let $U^j(t) = (v^j(t); u^j(t); u_t^j(t))$, $j = 1, 2$, be two different solutions to the problem in question with the same initial data. Then their difference $U(t) = U^1(t) - U^2(t) = (v(t); u(t); u_t(t))$ satisfies the variational equality

$$-\int_0^T (v, \phi_t)_{\mathcal{O}} + \nu \int_0^T (\nabla v, \nabla \phi)_{\mathcal{O}} - \int_0^T (u_t, \partial_t b)_{\Omega} + \int_0^T (\Delta u, \Delta b)_{\Omega} = 0 \quad (47)$$

for all $\phi \in \mathcal{L}_T$, $b = (\phi|_{\Omega})^3$. Now for every $0 < s < T$ we take

$$\phi(t) \equiv \phi^s(t) = \begin{cases} -\int_t^s d\tau \int_0^{\tau} d\zeta v(\zeta), & t < s, \\ 0, & t \geq s, \end{cases} \quad (48)$$

as a test function. We denote

$$\psi^s(t) = \partial_t \phi^s(t) = \int_0^t d\zeta v(\zeta) \quad \text{and} \quad b^s(t) = (\phi^s(t)|_{\Omega})^3 = -\int_t^s d\tau u(\tau).$$

Substituting ϕ^s into (47), we obtain

$$-\int_0^s (v, \psi^s)_{\mathcal{O}} + \nu \int_0^s (\nabla \partial_t \psi^s, \nabla \phi^s)_{\mathcal{O}} - \int_0^s (u_t, u)_{\Omega} + \int_0^s (\partial_t \Delta b^s, \Delta b^s)_{\Omega} = 0. \quad (49)$$

Integrating by parts the second term in (49) and using the relations $\psi^s(0) = 0$ and $\phi^s(s) = 0$, we have

$$\int_0^s (\nabla \partial_t \psi^s, \nabla \phi^s)_{\mathcal{O}} = (\nabla \phi_s, \nabla \psi_s)_{\mathcal{O}} \Big|_0^s - \int_0^s (\nabla \psi^s, \nabla \psi^s)_{\mathcal{O}} = -\int_0^s \|\nabla \psi^s\|_{\mathcal{O}}^2.$$

Therefore (49) yields

$$\|\psi^s(s)\|_{\mathcal{O}}^2 + 2\nu \int_0^s \|\nabla \psi^s\|_{\mathcal{O}}^2 dt + \|u(s)\|_{\Omega}^2 + \|\Delta b^s(0)\|_{\Omega}^2 = 0$$

for almost all $0 \leq s \leq T$. Therefore $v(s) = 0$ and $u(s) = 0$ for almost all $0 \leq s \leq T$. Thus the uniqueness is proved.

Step 5. Continuity with respect to t and the energy equality. First we note that any weak solution $(v(t); u(t); u_t(t))$ is weakly continuous in $X \times$

$H_0^2(\Omega) \times \widehat{L}_2(\Omega)$. Indeed, it follows from (24) that that any weak solution $(v(t); u(t))$ satisfies the relation

$$(v(t), \psi)_{\mathcal{O}} = (v_0, \psi(0))_{\mathcal{O}} + \int_0^t [-\nu(\nabla v, \nabla \psi)_{\mathcal{O}} + (G_f(\tau), \psi)_{\mathcal{O}}] d\tau$$

for almost all $t \in [0, T]$ and for all $\psi \in \widetilde{V} = \{v \in V : v|_{\Omega} = 0\} \subset W$. This implies that $v(t)$ is weakly continuous in \widetilde{V}' . Since $X \subset \widetilde{V}'$, we can apply the Lions lemma (see [27, Lemma 8.1]) and conclude that $v(t)$ is weakly continuous in X . The same lemma gives us weak continuity of $u(t)$ in $H_0^2(\Omega)$. Now using (24) again we conclude that $(u_t(t), \beta)_{\Omega}$ is continuous for $\beta \in \widehat{H}_0^2(\Omega)$. The density argument yields weak continuity of $u_t(t)$ in $\widehat{L}_2(\Omega)$.

To prove the energy equality, we follow the scheme of [28, Ch.1], see also [27, Ch.3]. We first note that due to Remark 3.4 it is sufficient to consider the case when $U_0 = (v_0; u_0; u_1) \in \mathcal{H}$. Then for every fixed $0 < s < t < T$ we introduce a piecewise-linear continuous function $\theta_n(\tau)$ on \mathbb{R} such that $\theta_n(\tau) = 1$ for $s \leq \tau \leq t$ and $\theta_n(\tau) = 0$ when $\tau < s - 1/n$ or $\tau > t + 1/n$. Let $\rho_k \in C_0^\infty(\mathbb{R})$ be an even function such that $\text{supp } \rho_k \subset [-k^{-1}, k^{-1}]$ and $\int_{\mathbb{R}} \rho_k(s) ds = 1$. Now for k and n large enough we consider the function $\phi(\tau) = \theta_n((\theta_n v) * \rho_k * \rho_k)$, where v is a weak solution to (17)–(21), as a test function in variational equality (22). Substituting this ϕ into (22) and passing to the limit when $k \rightarrow \infty$ we obtain that

$$\begin{aligned} - \int_0^T \theta_n \dot{\theta}_n [\|v\|_{\mathcal{O}}^2 + \|u_\tau\|_{\Omega}^2 + \|\Delta u\|_{\Omega}^2] d\tau + \nu \int_0^T \theta_n^2 \|\nabla v\|_{\mathcal{O}}^2 d\tau \\ = \int_0^T \theta_n^2 [(G_f(\tau), v)_{\mathcal{O}} + (G_{pl}(\tau), u_k)_{\Omega}] d\tau \end{aligned} \quad (50)$$

As in [28, Ch. 1] one can see that for every function $h \in L_1(0, T)$

$$\lim_{n \rightarrow \infty} \int_0^T \theta_n(\tau) \dot{\theta}_n(\tau) h(\tau) d\tau = -\frac{1}{2} [h(t) - h(s)]$$

for almost all s and t . Therefore after the limit transition in (50) we obtain energy relation (28) valid for almost all s and t . Now using weak continuity of the solution $(v(t); u(t))$ and the energy inequality (which is valid for $s = 0$ and for every t) we can establish the energy equality. As in [27, Ch. 3] this also implies strong continuity of weak solutions with respect to t .

Step 6. Exponential stability. To prove the exponential stability estimate in (30), we construct a Lyapunov function using an idea from [11]. Let

$$V(v_0, u_0, u_1) = \mathcal{E}_0(v_0, u_0, u_1) + \epsilon \Psi(v_0, u_0, u_1),$$

where $\Psi(v_0, u_0, u_1) = (u_0, u_1)_{\Omega} + (v_0, N_0 u_0)_{\mathcal{O}}$ with N_0 defined by (13), and $\epsilon > 0$ is a small parameter which will be chosen later. We consider these

functionals on approximate solutions $(v_{n,m}; u_n)$ for which $\hat{P}u_0 = u_0$ and thus $\hat{P}u_n(t) = u_n(t)$ for all $t > 0$. This allow us to substitute in (37) N_0u_n instead of χ and obtain that

$$\begin{aligned} \frac{d}{dt}\Psi_{n,m}(t) &\equiv \frac{d}{dt}\Psi(v_{n,m}(t), u_n(t), \partial_t u_n(t)) = \|\partial_t u_n\|_{\Omega}^2 + (v_{n,m}, N_0 \partial_t u_n)_{\mathcal{O}} \\ &\quad - \nu(\nabla v_{n,m}, \nabla N_0 u_n)_{\mathcal{O}} - \|\Delta u_n\|_{\Omega}^2 + (G_f, N_0 u_n)_{\mathcal{O}} + (G_{pl}, u_n)_{\Omega}. \end{aligned} \quad (51)$$

By Proposition 2.2, using the compatibility condition in (36) and the trace theorem we have that

$$|(v_{n,m}, N_0 \partial_t u_n)_{\mathcal{O}}| \leq C \|v_{n,m}\|_{\mathcal{O}} \|\partial_t u_n\|_{\Omega} \leq C \|\nabla v_{n,m}\|_{\mathcal{O}}^2.$$

Similarly,

$$|(\nabla v_{n,m}, \nabla N_0 u_n)_{\mathcal{O}}| \leq \eta \|\Delta u_n\|^2 + C_{\eta} \|\nabla v_{n,m}\|_{\mathcal{O}}^2, \quad \forall \eta > 0,$$

and also

$$|(G_f, N_0 u_n)_{\mathcal{O}} + (G_{pl}, u_n)_{\Omega}| \leq \eta \|\Delta u_n\|^2 + C_{\eta} \left[\|G_f\|_{V'}^2 + \|G_{pl}\|_{-1/2, \Omega}^2 \right].$$

Therefore it follows from (51) that

$$\frac{d}{dt}\Psi_{n,m}(t) \leq -\frac{1}{2}\|\Delta u_n\|^2 + C \|\nabla v_{n,m}\|_{\mathcal{O}}^2 + C \left[\|G_f\|_{V'}^2 + \|G_{pl}\|_{-1/2, \Omega}^2 \right].$$

Using the energy relation in (38) we also have that

$$\frac{d}{dt}\mathcal{E}_0(v_{n,m}(t), u_n(t), \partial_t u_n(t)) \leq -\frac{\nu}{2}\|\nabla v_{n,m}\|_{\mathcal{O}}^2 + C_{\nu} \left[\|G_f\|_{V'}^2 + \|G_{pl}\|_{-1/2, \Omega}^2 \right].$$

Therefore the function $V_{n,m}(t) \equiv V(v_{n,m}(t), u_n(t), \partial_t u_n(t))$ satisfies the relations

$$a_0 \mathcal{E}_0(v_{n,m}(t), u_n(t), \partial_t u_n(t)) \leq V_{n,m}(t) \leq a_1 \mathcal{E}_0(v_{n,m}(t), u_n(t), \partial_t u_n(t))$$

for sufficiently small $\varepsilon > 0$ and

$$\frac{d}{dt}V_{n,m}(t) + a_2 V_{n,m}(t) \leq a_3 \left[\|G_f\|_{V'}^2 + \|G_{pl}\|_{-1/2, \Omega}^2 \right]$$

with positive constants a_i . This implies relation (30) for approximate solutions. The limit transition yields (30) for every weak solutions.

This completes the proof of Theorem 3.3.

4 Nonlinear problem

In this section we deal with problem (1)–(6) with a nonlinear feedback force. First we describe hypotheses concerning this force. Then we prove well-posedness (see Theorem 4.3) and construct the corresponding semiflow. Our main result (see Theorem 4.8) states the existence of finite-dimensional attractor.

4.1 Structure of feedback force

We impose the following hypotheses concerning the nonlinear feedback force $\mathcal{F}(u)$ in the plate equation (4).

Assumption 4.1 (F1) *There exists $\epsilon > 0$ such that $\mathcal{F}(u)$ is locally Lipschitz from $H_0^{2-\epsilon}(\Omega)$ into $H^{-1/2}(\Omega)$ ¹ in the sense that*

$$\|\mathcal{F}(u_1) - \mathcal{F}(u_2)\|_{-1/2,\Omega} \leq C_R \|u_1 - u_2\|_{2-\epsilon,\Omega} \quad (52)$$

for any $u_i \in H_0^2(\Omega)$ such that $\|u_i\|_{2,\Omega} \leq R$.

(F2) *There exists a C^1 -functional $\Pi(u)$ on $H_0^2(\Omega)$ such that $\mathcal{F}(u) = \Pi'(u)$, where Π' denotes the Fréchet derivative of Π .*

(F3) *The plate force potential Π is bounded on bounded sets from $H_0^2(\Omega)$ and there exist $\eta < 1/2$ and $C \geq 0$ such that*

$$\eta \|\Delta u\|_{\Omega}^2 + \Pi(u) + C \geq 0, \quad \forall u \in H_0^2(\Omega). \quad (53)$$

The nonlinear feedback (elastic) force $\mathcal{F}(u)$ may have one of the following forms (which represent different plate models):

Kirchhoff model: $\mathcal{F}(u)$ is the Nemytskii operator

$$u \mapsto -\kappa \cdot \operatorname{div} \{ |\nabla u|^q \nabla u - \mu |\nabla u|^r \nabla u \} + f(u) - h(x),$$

where $\kappa \geq 0$, $q > r \geq 0$ and $\mu \in \mathbb{R}$ are parameters, $h \in L_2(\Omega)$, and

$$f \in \operatorname{Lip}_{\operatorname{loc}}(\mathbb{R}) \quad \text{satisfies} \quad \liminf_{|s| \rightarrow \infty} f(s)s^{-1} > -\lambda_1, \quad (54)$$

where λ_1 is the first eigenvalue of the biharmonic operator with the Dirichlet boundary conditions. In this case the relation in (52) follows from the considerations given in [12, Sect.5]. We also have that

$$\begin{aligned} \Pi(u) = & \int_{\Omega} F(u(x)) dx + \frac{\kappa}{q+2} \int_{\Omega} |\nabla u(x)|^{q+2} dx \\ & - \frac{\kappa\mu}{r+2} \int_{\Omega} |\nabla u(x)|^{r+2} dx - \int_{\Omega} u(x)h(x) dx, \end{aligned}$$

where $F(s) = \int_0^s f(\xi) d\xi$ is the antiderivative of f . Due to the second relation in (54) we obviously have (53).

Von Karman model: This model is well known in nonlinear elasticity and constitute a basic model describing nonlinear oscillations of a plate accounting for large deflections, see [28, 15] and the references therein. The

¹ We recall that according our definitions $H^{-1/2}(\Omega) = [H_*^{1/2}(\Omega)]' \not\subseteq [H_0^{1/2}(\Omega)]'$.

force \mathcal{F} has the form $\mathcal{F}(u) = -[u, v(u) + F_0] - h(x)$, where $F_0 \in H^4(\Omega)$ and $h \in L_2(\Omega)$ are given functions, the von Karman bracket $[u, v]$ is given by

$$[u, v] = \partial_{x_1}^2 u \cdot \partial_{x_2}^2 v + \partial_{x_2}^2 u \cdot \partial_{x_1}^2 v - 2 \cdot \partial_{x_1 x_2}^2 u \cdot \partial_{x_1 x_2}^2 v,$$

and the Airy stress function $v(u)$ solves the following elliptic problem

$$\Delta^2 v(u) + [u, u] = 0 \quad \text{in } \Omega, \quad \frac{\partial v(u)}{\partial n} = v(u) = 0 \quad \text{on } \partial\Omega.$$

It is known (see, e.g., Corollary 1.4.5 in [15]) that

$$\|[u_1, v(u_1)] - [u_2, v(u_2)]\|_{-\eta, \Omega} \leq C(\|u_1\|_{2, \Omega}^2 + \|u_2\|_{2, \Omega}^2)\|u_1 - u_2\|_{2-\eta, \Omega}$$

for every $\eta \in [0, 1]$, which implies (52). The potential Π has the form

$$\Pi(u) = \frac{1}{4} \int_{\Omega} [|v(u)|^2 - 2([u, F_0] - 2h)u] \, dx$$

and possesses the properties listed in Assumption 4.1, see, e.g., [15, Chapter 4] for details.

Berger Model: In this case the feedback force has the form

$$\mathcal{F}(u) = - \left[\kappa \int_{\Omega} |\nabla u|^2 dx - \Gamma \right] \Delta u - h(x),$$

where $\kappa > 0$ and $\Gamma \in \mathbb{R}$ are parameters, $h \in L_2(\Omega)$. One can see Assumption 4.1 is satisfied, for some details and references see, e.g., [10, Chapter 4] and [14, Chapter 7].

4.2 Well-Possedness

Definition 4.2 A pair of functions $(v(t); u(t))$ is said to be a weak solution to (1)–(6) on a time interval $[0, T]$ if

- $v \in L_{\infty}(0, T; X) \cap L_2(0, T; V)$;
- $u \in L_{\infty}(0, T; H_0^2(\Omega))$, $u_t \in L_{\infty}(0, T; \widehat{L}_2(\Omega))$, $u(0) = u_0$;
- the equality in (22) holds with $G_{pl}(t) := -\mathcal{F}(u(t)) + G_{pl}(t)$;
- the compatibility condition $v(t)|_{\Omega} = (0; 0; u_t(t))$ holds for almost all t .

Theorem 4.3 Assume that $U_0 = (v_0; u_0; u_1) \in \mathcal{H}$, $G_f(t) \in L_2(0, T; V')$ and $G_{pl}(t) \in L_2(0, T; H^{-1/2}(\Omega))$. Then for any interval $[0, T]$ there exists a unique weak solution $(v(t); u(t))$ to (1)–(6) with the initial data U_0 . This solution possesses the property

$$U(t) \equiv (v(t); u(t); u_t(t)) \in C(0, T; \mathcal{H}), \quad (55)$$

where \mathcal{H} is given by (25), and satisfies the energy balance equality

$$\begin{aligned} \mathcal{E}(v(t), u(t), u_t(t)) + \nu \int_0^t \|\nabla v\|_{\mathcal{O}}^2 d\tau &= \mathcal{E}(v_0, u_0, u_1) \\ &+ \int_0^t (G_f(\tau), v)_{\mathcal{O}} d\tau + \int_0^t (G_{pl}(\tau), u_\tau)_{\Omega} d\tau \end{aligned} \quad (56)$$

for every $t > 0$, where the energy functional \mathcal{E} is defined by the relation

$$\mathcal{E}(v, u, u_t) = \frac{1}{2} \|v\|_{\mathcal{O}}^2 + E(u, u_t)$$

with the plate energy $E(u, u_t)$ given by

$$E(u, u_t) = \frac{1}{2} (\|u_t\|_{\Omega}^2 + \|\Delta u\|_{\Omega}^2) + \int_{\Omega} \Pi(u(x)) dx.$$

Moreover, there exists a constant $a_{R,T} > 0$ such that for any couple of weak solutions $U(t) = (v(t); u(t); u_t(t))$ and $\hat{U}(t) = (\hat{v}(t); \hat{u}(t); \hat{u}_t(t))$ with the initial data possessing the property $\|U_0\|_{\mathcal{H}}, \|\hat{U}_0\|_{\mathcal{H}} \leq R$ we have

$$\|U(t) - \hat{U}(t)\|_{\mathcal{H}}^2 + \int_0^t \|\nabla(v - \hat{v})\|_{\mathcal{O}}^2 d\tau \leq a_{R,T} \|U_0 - \hat{U}_0\|_{\mathcal{H}}^2, \quad t \in [0, T]. \quad (57)$$

The spatial average of $u(t)$ is preserved. In particular, if $U_0 \in \widehat{\mathcal{H}}$, then $U(t) \in \widehat{\mathcal{H}}$ for every $t > 0$. We recall that $\widehat{\mathcal{H}}$ is defined by (26).

Proof. The proof of the local existence of an approximate solution is almost the same, as in the linear case (see Theorem 3.3). We use approximate solutions of the same structure as in (31) which satisfy (32), (34) and also (33) with $-\mathcal{F}(u_n(t)) + G_{pl}(t)$ instead of $G_{pl}(t)$. Then using the standard argument we establish the energy relation in (56) for these approximate solutions. Now the positivity type estimate in (53) allow us to obtain the same a priori estimates as in (39) and (40). Therefore we can prove the global existence of approximate solutions and establish the existence of a weak solution $U(t) = (v(t); u(t); u_t(t))$ by the same argument as in the linear case. To make limit transition in the nonlinear term we use (52).

Now we can consider the pair $(v(t); u(t))$ as a solution to linear problem with $G_{pl}(t) := -\mathcal{F}(u(t)) + G_{pl}(t)$. This allow us to obtain (55) and also derive energy balance relation (56) from (28) using the potential structure of the force \mathcal{F} : $\mathcal{F}(u) = \Pi'(u)$.

Since the difference of two weak solution can be treated as a solution to the linear problem with $G_f \equiv 0$ and $G_{pl}(t) := \mathcal{F}(\hat{u}(t)) - \mathcal{F}(u(t))$, we can obtain (57) from the energy equality (28). The uniqueness follows from (57).

Preservation of the spatial average of $u(t)$ follows from the same property for approximate solutions. \square

Remark 4.4 In the autonomous case we can suggest another form of energy relation (56). Let $G_{pl}(t) \equiv 0$ and $G_f(t) \equiv G_0 \in V'$ be independent of t . Suppose that a pair $(v_*; p_*) \in V \times L^2(\mathcal{O})$ solve problem (10) with $g \equiv G_0$ and $\psi \equiv 0$, i.e.,

$$-\nu \Delta v_* + \nabla p_* = G_0, \quad \operatorname{div} v_* = 0 \quad \text{in } \mathcal{O}; \quad v_* = 0 \quad \text{on } \partial\mathcal{O}. \quad (58)$$

Then the following form of the energy balance equation is valid:

$$\mathcal{E}_*(v(t), u(t), u_t(t)) + \nu \int_0^t \|\nabla(v - v_*)\|_{\mathcal{O}}^2 d\tau = \mathcal{E}_*(v_0, u_0, u_1), \quad (59)$$

where

$$\mathcal{E}_*(v, u, u_t) = \frac{1}{2} \|v - v_*\|_{\mathcal{O}}^2 + E_*(u, u_t)$$

with $E_*(u, u_t)$ given by

$$E_*(u, u_t) = \frac{1}{2} (\|u_t\|_{\Omega}^2 + \|\Delta u\|_{\Omega}^2) + \int_{\Omega} \Pi(u(x)) dx - (p_*, u)_{\Omega}.$$

Indeed, it follows from (58) that

$$(G_0, v(t))_{\mathcal{O}} = \nu(\nabla v_*, \nabla v(t))_{\mathcal{O}} + \frac{d}{dt}(p_*, u(t))_{\Omega}.$$

Substituting $\psi = v_*$ in (24) we also have that

$$\frac{d}{dt}(v(t), v_*)_{\mathcal{O}} + \nu(\nabla v_*, \nabla v(t)) = (G_0, v_*)_{\mathcal{O}} = \nu \|\nabla v_*\|_{\mathcal{O}}^2.$$

Therefore

$$(G_0, v(t))_{\mathcal{O}} = \frac{d}{dt} [(v(t), v_*)_{\mathcal{O}} + (p_*, u(t))_{\Omega}] + 2\nu(\nabla v_*, \nabla v(t)) - \nu \|\nabla v_*\|_{\mathcal{O}}^2.$$

This and also the energy relation in (56) imply (59).

This remark allows us to derive from Theorem 4.3 the following assertion.

Corollary 4.5 *Let $G_f(t) \equiv G_0 \in V'$ be independent of t and $G_{pl}(t) \equiv 0$. Then problem (1)–(6) generates dynamical systems (S_t, \mathcal{H}) and $(S_t, \widehat{\mathcal{H}})$ with the evolution operator defined by the formula $S_t U_0 = (v(t); u(t); u_t(t))$, where $(v; u)$ is a weak solution to (1)–(6) with the initial data $U_0 = (v_0; u_0; u_1)$. These systems are gradient with the full energy $\mathcal{E}_*(v_0, u_0, u_1)$ as a Lyapunov function. This means that (a) $U \mapsto \mathcal{E}_*(U)$ is continuous on \mathcal{H} , (b) $\mathcal{E}_*(S_t U_0)$ is not increasing in t , and (c) if $\mathcal{E}_*(S_t U_0) = \mathcal{E}_*(U_0)$ for some $t > 0$, then U_0 is a stationary point of S_t (i.e., $S_t U_0 = U_0$ for all $t \geq 0$). Moreover, the set $\mathcal{E}_R = \{U_0 : \mathcal{E}_*(U_0) \leq R\}$ is a bounded closed forward invariant set for every $R > 0$.*

Proof. We need only to check the properties of the functional \mathcal{E}_* .

It is clear from Assumption 4.1(F2) that \mathcal{E}_* is continuous on \mathcal{H} .

By (59) We have that $\mathcal{E}_*(S_t U_0) \leq \mathcal{E}_*(S_\tau U_0)$ for $t \geq \tau \geq 0$. This gives the monotonicity of $t \mapsto \mathcal{E}_*(S_t U_0)$ and the invariance of \mathcal{E}_R .

If $\mathcal{E}_*(S_{t_0} U_0) = \mathcal{E}_*(U_0)$ for some $t_0 > 0$, then (59) implies that $v(t) = v_*$ for all $t \in [0, t_0]$ and thus $u_t(t) = v_*^3|_\Omega = 0$. Hence $u(t) \equiv u$ for some $u \in H_0^2(\Omega)$ and $U_0 = (v_*; u; 0)$ is a stationary point for S_t . \square

Below we describe the set of stationary point of the evolution semigroup S_t with more details.

4.3 Stationary solutions

As above we assume that $G_{pl} \equiv 0$ and $G_f(t) \equiv G_0 \in V'$ is independent of t . Let $\tilde{V} = \{u \in V : v|_{\partial\mathcal{O}} = 0\}$. It follows from Definition 4.2 that a stationary (time-independent) solution is a pair $(v; u)$ from $\tilde{V} \times H_0^2(\Omega)$ satisfying the relation

$$\nu(\nabla v, \nabla \psi)_{\mathcal{O}} + (\Delta u, \Delta \beta)_{\Omega} - (G_0, \psi)_{\mathcal{O}} + (\mathcal{F}(u), \beta)_{\Omega} = 0 \quad (60)$$

for any $\psi \in W$ with $\psi^3|_\Omega = \beta$, where W is given by (23). Using (59) we have that $v = v_*$, where v_* solves (58). One can also see $(\nabla v, \nabla N_0 \beta)_{\mathcal{O}} = 0$ for any $v \in V_0$ and $\beta \in \widehat{H}_0^2(\Omega)$, where N_0 is defined in (13). Therefore from (60) with $\psi = N_0 \beta$ we have the following variational problem for $u \in H_0^2(\Omega)$:

$$(\Delta u, \Delta \beta)_{\Omega} + (\mathcal{F}(u) - N_0^* G_0, \beta)_{\Omega} = 0, \quad \forall \beta \in \widehat{H}_0^2(\Omega). \quad (61)$$

The following calculation performed first on smooth functions gives us

$$\begin{aligned} (G_0, N_0 \beta)_{\mathcal{O}} &= (-\nu \Delta v_* + \nabla p_*, N_0 \beta)_{\mathcal{O}} \\ &= (v_*, -\nu \Delta N_0 \beta)_{\mathcal{O}} + (p_*, \beta)_{\Omega} = (v_*, -\nabla p_\beta)_{\mathcal{O}} + (p_*, \beta)_{\Omega} = (p_*, \beta)_{\Omega}. \end{aligned}$$

Since the pressure p_* in (58) is defined up to a constant, we can suppose that $p_* = N_0^* G_0$. By Proposition 2.2 $N_0^* : V' \mapsto [\widehat{H}_*^{1/2}(\Omega)]'$. This provides us with the regularity of the pressure impact on the plate.

One can see that a function $u \in H_0^2(\Omega)$ solves (61) if and only if u is a variational solution to problem

$$\Delta^2 u + \mathcal{F}(u) - p_* = C \text{ in } \Omega, \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \quad (62)$$

for some constant C which may depend on u . Since every variational solution to (62) is an extreme point of the functional

$$\Psi(u) = \frac{1}{2} \|\Delta u\|_{\Omega}^2 + \int_{\Omega} \Pi(u(x)) dx - (p_* + C, u)_{\Omega},$$

using relation (53) in Assumption 4.1 we can prove the existence of these solutions. Thus we obtain a family of solutions to (60) parameterized by the real parameter C . To fix somehow the constant C in (62) it is convenient to fix the average of u . In the case of the zero average we obtain the following assertion.

Proposition 4.6 *In addition to Assumption 4.1 we assume that $G_0 \in V'$ and there exist $\eta < 1/2$ and $c \geq 0$ such that*

$$\eta \|\Delta u\|_{\Omega}^2 + (u, \mathcal{F}(u))_{\Omega} \geq -c, \quad \forall u \in H_0^2(\Omega). \quad (63)$$

Then the set \mathcal{N}_0 of solutions u to problem (61) with the property $\int_{\Omega} u dx = 0$ is nonempty compact set in $\hat{H}_0^2(\Omega)$.

Proof. Restricting the functional Ψ on $\hat{H}_0^2(\Omega)$ we can prove the existence of its minimum point on $\hat{H}_0^2(\Omega)$. This means that \mathcal{N}_0 is not empty. If $u \in \hat{H}_0^2(\Omega)$ is a solution, then taking $\beta = u$ in (61) and using (63) we conclude that \mathcal{N}_0 is bounded in $H_0^2(\Omega)$. If $\{u_n\}$ is a sequence from \mathcal{N}_0 , then from (61) we conclude that

$$\|\Delta(u_n - u_m)\|_{\Omega}^2 \leq C \|\mathcal{F}(u_n) - \mathcal{F}(u_m)\|_{-1/2, \Omega} \|u_n - u_m\|_{1/2, \Omega}.$$

Thus (52) yields $\|\Delta(u_n - u_m)\|_{\Omega} \leq C \|u_n - u_m\|_{2-\varepsilon, \Omega}$. This implies that the sequence $\{u_n\}$ is relatively compact. \square

Remark 4.7 A similar result can be obtain for the set \mathcal{N}_{α} of solutions u to problem (61) with the property $\langle u \rangle \equiv \int_{\Omega} u dx' = \alpha$ with a fixed $\alpha \in \mathbb{R}$, if instead of (63) we assume that there exist $\eta < 1/2$, $c_{\alpha} \geq 0$ and a smooth function ϕ with the property $\langle \phi \rangle = \alpha$ such that

$$\eta \|\Delta u\|_{\Omega}^2 + (u, \mathcal{F}(u))_{\Omega} - (\phi, \mathcal{F}(u))_{\Omega} \geq -c_{\alpha}, \quad \forall u \in H_0^2(\Omega). \quad (64)$$

Indeed, if we consider the functional Ψ on $\hat{H}_{0,\alpha}^2 = \{u \in H_0^2(\Omega) : \langle u \rangle = \alpha\}$ for some fixed constant C , then we can prove the existence of a solution u to (61) in $\hat{H}_{0,\alpha}^2$. Now substituting $\beta = u - \phi$ in (61) and using (64) we obtain the boundedness of the set \mathcal{N}_{α} in $\hat{H}_{0,\alpha}^2$. To prove the compactness of \mathcal{N}_{α} we use the same argument as in Proposition 4.6.

It follows from Proposition 4.6 that the set of all stationary points of S_t in the space $\hat{\mathcal{H}}$ is nonempty compact set and has the form

$$\mathcal{N} = \left\{ (v_*; u; 0) : (v_*; u) \in V_0 \times \hat{H}_0^2(\Omega) \text{ solve (58) and (61)} \right\} \quad (65)$$

4.4 Asymptotical behavior

In this section we are interested in global asymptotic behavior of the dynamical system $(S_t, \widehat{\mathcal{H}})$. Our main result states the existence of a compact global attractor of finite fractal dimension.

We recall (see, e.g., [5, 10, 31]) that the *global attractor* of the dynamical system $(S_t, \widehat{\mathcal{H}})$ is defined as a bounded closed set $\mathfrak{A} \subset \widehat{\mathcal{H}}$ which is invariant ($S(t)\mathfrak{A} = \mathfrak{A}$ for all $t > 0$) and uniformly attracts all other bounded sets:

$$\lim_{t \rightarrow \infty} \sup \{ \text{dist}_{\mathcal{H}}(S(t)y, \mathfrak{A}) : y \in B \} = 0 \quad \text{for any bounded set } B \text{ in } \widehat{\mathcal{H}}.$$

The *fractal dimension* $\dim_f^X M$ of a compact set M in a complete metric space X is defined as

$$\dim_f^X M = \limsup_{\varepsilon \rightarrow 0} \frac{\ln N(M, \varepsilon)}{\ln(1/\varepsilon)},$$

where $N(M, \varepsilon)$ is the minimal number of closed sets in X of diameter 2ε which cover M .

We also recall (see, e.g., [5]) that the *unstable set* $\mathbb{M}_+(\mathcal{N})$ emanating from some set $\mathcal{N} \subset \widehat{\mathcal{H}}$ is a subset of $\widehat{\mathcal{H}}$ such that for each $z \in \mathbb{M}_+(\mathcal{N})$ there exists a full trajectory $\{y(t) : t \in \mathbb{R}\}$ satisfying $y(0) = z$ and $\text{dist}(y(t), \mathcal{N}) \rightarrow 0$ as $t \rightarrow -\infty$.

Theorem 4.8 *Let Assumption 4.1 be in force. Assume that $G_f(t) \equiv G_0 \in V'$ is independent of t , $G_{pl}(t) \equiv 0$ and (63) holds. Then the dynamical system $(S_t, \widehat{\mathcal{H}})$ possesses a compact global attractor \mathfrak{A} . Moreover,*

- (1) $\mathfrak{A} = \mathbb{M}_+(\mathcal{N})$, where \mathcal{N} is the set of equilibria given by (65).
- (2) This attractor has a finite fractal dimension in $\widehat{\mathcal{H}}$.
- (3) Any trajectory $\gamma = \{(v(t); u(t); u_t(t)) : t \in \mathbb{R}\}$ from the attractor \mathfrak{A} possesses the properties

$$(v_t; u_t; u_{tt}) \in L_\infty(\mathbb{R}; X \times \widehat{H}_0^2(\Omega) \times \widehat{L}_2(\Omega)) \quad (66)$$

and there is $R > 0$ such that

$$\sup_{\gamma \subset \mathfrak{A}} \sup_{t \in \mathbb{R}} (\|v_t\|_{\mathcal{O}}^2 + \|u_t\|_{2, \Omega}^2 + \|u_{tt}\|_{\Omega}^2) \leq R^2. \quad (67)$$

Remark 4.9 We cannot state a similar result on the existence of a global attractor for the system (S_t, \mathcal{H}) . The point is that the average of $u(t)$ is preserved and thus the system (S_t, \mathcal{H}) is non-dissipative. However using the same procedure as for the linear case (see Remark 3.4) we can study the long-time behavior of (S_t, \mathcal{H}) by means of a family of dissipative problems

in $\widehat{\mathcal{H}}$. Indeed, we can decompose the solution to (1)–(6) with the initial data $(v_0; u_0; u_1)$ into the sum $(v(t); u(t); u_t(t)) = (\bar{v}(t); \bar{u}(t); \bar{u}_t(t)) + (0; \psi; 0)$, where $\psi = (I - \widehat{P})u_0$ and $(\bar{v}(t); \bar{u}(t); \bar{u}_t(t))$ solves (1)–(3), (5), (6) with the plate equation

$$\bar{u}_{tt} + \Delta^2 \bar{u} + \mathcal{F}(\bar{u} + \psi) + \Delta^2 \psi = G_{pl}(t) + p|_{\Omega}$$

(instead of (4)) and with the initial conditions $(v_0, \widehat{P}u_0, u_1)$.

To obtain the result stated in Theorem 4.8 it is sufficient to show that the system is quasi-stable (in the sense of [15]). For this we use the stability properties of linear problem (17)–(21) established in Theorem 3.3 to prove the following assertion.

Lemma 4.10 (Quasi-stability) *Let $U^i(t) = (v^i(t); u^i(t); u_t^i(t))$, $i = 1, 2$, be two weak solutions with initial data $U_0^i = (v_0^i; u_0^i; u_1^i)$ from \mathcal{H} such that $\|U_0^i\|_{\mathcal{H}} \leq R$, $i = 1, 2$, then their difference*

$$Z(t) = U^1(t) - U^2(t) \equiv (v(t); u(t); u_t(t))$$

satisfies the relation

$$\|Z(t)\|_{\mathcal{H}}^2 \leq M_R e^{-\gamma_* t} \|Z_0\|_{\mathcal{H}}^2 + M_R \int_0^t e^{-\gamma_*(t-\tau)} \|u(\tau)\|_{\Omega}^2 d\tau \quad (68)$$

for some positive constant M_R and γ_ .*

Proof. We consider $(v(t); u(t))$ as a solution to linear problem (17)–(20) with $G_f \equiv 0$ and $G_{pl}(t) = -\mathcal{F}(u^1(t)) + \mathcal{F}(u^2(t))$. Therefore it follows from (52) and (30) that

$$\|Z(t)\|_{\mathcal{H}}^2 \leq M e^{-\gamma t} \|Z_0\|_{\mathcal{H}}^2 + C_R \int_0^t e^{-\gamma(t-\tau)} \|u(\tau)\|_{2-\varepsilon, \Omega}^2 d\tau.$$

Hence the interpolation relation

$$\|u\|_{2-\varepsilon, \Omega}^2 \leq \eta \|Z\|_{\mathcal{H}}^2 + c_{\eta} \|u\|_{\Omega}^2, \quad \forall \eta > 0,$$

via Gronwall's type argument, implies the conclusion in (68). \square

Proof of Theorem 4.8

Lemma 4.10 means that the dynamical system $(S_t, \widehat{\mathcal{H}})$ is quasi-stable in the sense of Definition 7.9.2 [15]. Therefore by Proposition 7.9.4 [15] $(S_t, \widehat{\mathcal{H}})$ is asymptotically smooth. Since the system is gradient, the boundedness of the set of the stationary points implies that there exists a compact global attractor. Moreover, the standard results on gradient systems with compact attractors (see, e.g., [5, 10, 31]) give us that $\mathfrak{A} = \mathbb{M}_+(\mathcal{N})$.

Since $(S_t, \widehat{\mathcal{H}})$ is quasi-stable the finiteness of fractal dimension $\dim_f \mathfrak{A}$ follows from Theorem 7.9.6 [15].

To obtain the result on regularity stated in (66) and (67) we apply Theorem 7.9.8 [15].

A Appendix: Generator of linear semigroup

To find the structure of the semigroup T_t generated by (17)–(21) in the space $\widehat{\mathcal{H}}$ we note that the evolution problem in (24) with $G_f \equiv 0$ and $G_{pl} \equiv 0$ can be written in the form

$$\frac{d}{dt} [(v, \psi)_{\mathcal{O}} + (u(t), \chi)_{\Omega} + (w(t), \beta)_{\Omega}] + \mathcal{A}(U(t), \Psi) = 0,$$

where $U = (v; u; w)$ is an element from $C(\mathbb{R}_+; \widehat{\mathcal{H}})$ with $v \in L_2^{loc}(\mathbb{R}_+; V)$ and $v|_{\Omega} = (0; 0; w)$. The text function $\Psi = (\psi; \chi; \beta)$ belongs to the space

$$\mathcal{W} \equiv \left\{ (\psi; \chi; \beta) \in W \times \widehat{H}_0^2(\Omega) \times \widehat{H}_0^2(\Omega) : \psi|_{\Omega} = (0; 0; \beta) \right\} \subset \widehat{\mathcal{H}},$$

and the bilinear form $\mathcal{A}(U, \Psi)$ is defined by the relation

$$\mathcal{A}(U, \Psi) = \nu(\nabla v, \nabla \psi)_{\mathcal{O}} - (w, \chi)_{\Omega} + (\Delta u, \Delta \beta)_{\Omega}.$$

Thus to describe the domain of the generator we need to describe all elements $U = (v; u; w)$ from

$$\mathcal{V} \equiv \left\{ (v; u; w) \in V \times \widehat{H}_0^2(\Omega) \times \widehat{L}_2(\Omega) : v|_{\Omega} = (0; 0; w) \right\} \subset \widehat{\mathcal{H}}$$

which solve the variational equation of the form

$$\mathcal{A}(U, \Psi) = (v, \psi)_{\mathcal{O}} + (u, \chi)_{\Omega} + (w, \beta)_{\Omega}, \quad \forall \Psi = (\psi; \chi; \beta) \in \mathcal{W},$$

where $F = (f_0; f_1; f_2)$ is a given element from $\widehat{\mathcal{H}}$. Taking $\psi \equiv 0$ one can see that $f_1 = -w \in \widehat{H}_0^2(\Omega)$. Therefore we arrive at the relation

$$\nu(\nabla v, \nabla \psi)_{\mathcal{O}} + (\Delta u, \Delta \beta)_{\Omega} = (f_0, \psi)_{\mathcal{O}} + (f_2, \beta)_{\Omega}. \quad (69)$$

By Proposition 2.2 we have that $N_0 w \in V \cap [H^2(\mathcal{O})]^3$ and the corresponding pressure p_w (defined by (13)) belongs to the class $H^1(\mathcal{O})/\mathbb{R}$. Since

$$\nu(\nabla N_0 w, \nabla \psi)_{\mathcal{O}} = -\nu(\Delta N_0 w, \psi)_{\mathcal{O}} = -(\nabla p_w, \psi)_{\mathcal{O}} = -(p_w, \beta)_{\Omega}.$$

We can rewrite (69) in the form

$$\nu(\nabla[v - N_0 w], \nabla \psi)_{\mathcal{O}} + (\Delta u, \Delta \beta)_{\Omega} = (f_0, \psi)_{\mathcal{O}} + (f_2 + p_w, \beta)_{\Omega} \quad (70)$$

for any $\psi \in W$. If we take now $\psi \in \widetilde{V} = \{v \in V : v|_{\partial \mathcal{O}} = 0\}$, then we obtain that $\tilde{v} = v - N_0 w \in \widetilde{V}$ solve the problem

$$-\nu \Delta \tilde{v} + \nabla p = f_0, \quad \text{div } \tilde{v} = 0 \quad \text{in } \mathcal{O}; \quad \tilde{v} = 0 \quad \text{on } \partial \mathcal{O}.$$

Since $f_0 \in X$, this implies that $\tilde{v} \in \widetilde{V} \cap [H^2(\mathcal{O})]^3$ and thus $v \in V \cap [H^2(\mathcal{O})]^3$. Therefore from (69) we have that

$$(P_S[-\nu \Delta v] - f_0, \psi)_{\mathcal{O}} + (\Delta u, \Delta \beta)_{\Omega} = (f_2, \beta)_{\Omega} \quad (71)$$

for every $\psi \in X$ with $\psi^3|_\Omega = \beta \in \widehat{H}_0^2(\Omega)$, where P_S is the orthoprojector in $[L_2(\mathcal{O})]^3$ on X . This implies that

$$P_S[-\nu\Delta v] - f_0 \perp \tilde{X} = \{u \in X : (u, n) = 0 \text{ on } \partial\mathcal{O}\}$$

Therefore (see, e.g., (2.70) in [4]) there exists $q \in H^1(\mathcal{O})$ such that

$$P_S[-\nu\Delta v] - f_0 = -\nabla q, \quad \Delta q = 0 \text{ in } \mathcal{O}, \quad \frac{\partial q}{\partial n}\Big|_S = 0. \quad (72)$$

Substitution in (71) yields $(\Delta u, \Delta\beta)_\Omega = (f_2 + q, \beta)_\Omega$ which implies that $u \in (H^4 \cap H_0^2)(\Omega)$. On the other hand, if we take $\psi = N_0\beta$ in (70), then due to the relation $(\nabla[v - N_0w], \nabla N_0\beta)_\mathcal{O} = 0$ we obtain

$$(\Delta u, \Delta\beta)_\Omega = (N_0^*f_0 + f_2 + p_w, \beta)_\Omega, \quad \forall \beta \in \widehat{H}_0^2(\Omega).$$

Thus, since the function q is defined up to a constant, we can suppose that

$$q|_\Omega = N_0^*f_0 + p_w = -f_2 + \Delta^2 u - \int_\Omega \Delta^2 u dx' \in \widehat{L}_2(\Omega). \quad (73)$$

Let us denote by $\mathcal{G} : H_*^{1/2}(\Omega) \mapsto X$ the mapping $r \mapsto \nabla q$, where $q \in H^1(\mathcal{O})$ solve the problem

$$\Delta q = 0 \text{ in } \mathcal{O}, \quad \frac{\partial q}{\partial n}\Big|_S = 0, \quad q|_\Omega = r.$$

Let

$$\bar{X} = \{u \in X : \gamma_n u \equiv (u, n)|_\Omega \in L_2(\Omega)\}$$

equipped with the graph norm $\|u\|_{\bar{X}}^2 = \|u\|_\mathcal{O}^2 + \|\gamma_n u\|_\Omega^2$. It is obvious that the trace operator γ_n is bounded from \bar{X} into $L^2(\Omega)$. One can see from calculations on smooth functions that

$$(\mathcal{G}r, \psi)_\mathcal{O} = \int_{\partial\mathcal{O}} q(\psi, n) dS = \int_\Omega r(\psi, n) dx', \quad \forall \psi \in \bar{X}.$$

and therefore

$$(\mathcal{G}\gamma_n \phi, \psi)_\mathcal{O} = (\gamma_n \phi, \gamma_n \psi)_\Omega, \quad \forall \phi, \psi \in \bar{X}. \quad (74)$$

Consequently the operator $\Gamma = \mathcal{G}\gamma_n$ can be extended to a bounded operator on \bar{X} . Moreover, by (74) Γ is nonnegative. With this operator Γ using the fact that that $f_2 = \gamma_n f_0$ we can write (72) in the form

$$f_0 + \Gamma f_0 = P_S[-\nu\Delta v] + \mathcal{G} \left[\Delta^2 u - \int_\Omega \Delta^2 u dx' \right] \equiv \mathcal{Q}(v, u).$$

This leads to the following description of the generator \mathcal{A} :

$$\mathcal{D}(\mathcal{A}) = \left\{ (v; u; w) \in \widehat{\mathcal{H}} \mid \begin{array}{l} v \in V \cap [H^2(\mathcal{O})]^3, u \in H^4(\Omega), \\ w \in \widehat{H}_0^2(\Omega), \gamma_n[\mathcal{Q}(v, u)] \in L_2(\Omega) \end{array} \right\}$$

and

$$\mathcal{A} \begin{bmatrix} v \\ u \\ w \end{bmatrix} = \begin{bmatrix} (1 + \Gamma)^{-1} \mathcal{Q}(v, u) \\ -w \\ \Delta^2 u - \int_{\Omega} \Delta^2 u dx' - p_w - N_0^* (1 + \Gamma)^{-1} \mathcal{Q}(v, u) \end{bmatrix}.$$

We can also write the operator \mathcal{A} in the form

$$\mathcal{A} \begin{bmatrix} v \\ u \\ w \end{bmatrix} = \begin{bmatrix} (1 + \Gamma)^{-1} \mathcal{Q}(v, u) \\ -w \\ \gamma_n (1 + \Gamma)^{-1} \mathcal{Q}(v, u) \end{bmatrix}.$$

References

- [1] G. Avalos, The strong stability and instability of a fluid-structure semigroup, *Appl. Math. Optim.*, **55** (2007), 163–184.
- [2] G. Avalos, R. Triggiani, The coupled PDE system arising in fluid–structure interaction. I. Explicit semigroup generator and its spectral properties, in: *Fluids and Waves*, Contemp. Math., vol. 440, AMS, Providence, RI, 2007, 15–54.
- [3] G. Avalos and R. Triggiani, Semigroup well-posedness in the energy space of a parabolic hyperbolic coupled Stokes–Lamé PDE system of fluid-structure interaction, *Discr. Contin. Dyn. Sys.*, Ser.S, **2** (2009), 417–447.
- [4] T. Azizov, V. Hardt, N. Kopachevsky, R. Mennicken, On the problem of small motions and normal oscillations of a viscous fluid in a partially filled container, *Math. Nachr.* **248-249** (2003), 3–39.
- [5] A.V. Babin, M.I. Vishik, *Attractors of Evolution Equations*. North-Holland, Amsterdam, 1992.
- [6] V. Barbu, Z. Grujić, I. Lasiecka, A. Tuffaha, Existence of the energy-level weak solutions for a nonlinear fluid–structure interaction model, in: *Fluids and Waves*, Contemp. Math., vol. 440, AMS, Providence, RI, 2007, 55–82.
- [7] V. Barbu, Z. Grujić, I. Lasiecka, A. Tuffaha, Smoothness of weak solutions to a nonlinear fluid–structure interaction model, *Indiana Univ. Math. J.* **57** (2008), 1173–207.
- [8] V.V. Bolotin, *Nonconservative Problems of Elastic Stability*, Pergamon Press, Oxford, 1963.
- [9] A. Chambolle, B. Desjardins, M. Esteban, C. Grandmont, Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate. *J. Math. Fluid Mech.* **7** (2005), 368–404.

- [10] I. Chueshov, *Introduction to the Theory of Infinite-Dimensional Dissipative Systems*. Acta, Kharkov, 1999 (in Russian); English translation: Acta, Kharkov, 2002; <http://www.emis.de/monographs/Chueshov/>.
- [11] I. Chueshov, A global attractor for a fluid-plate interaction model accounting only for longitudinal deformations of the plate, *Math. Methods Appl. Sci.* **34**, 1801–1812.
- [12] I. Chueshov, S. Kolbasin, Long-time dynamics in plate models with strong nonlinear damping, Preprint arXiv:1010.4991 (October 2010).
- [13] I. Chueshov and I. Lasiecka, Attractors for second order evolution equations, *J. Dynam. Diff. Eqs.*, 16 (2004), 469–512.
- [14] I. Chueshov and I. Lasiecka, *Long-Time Behavior of Second Order Evolution Equations with Nonlinear Damping*, Memoirs of AMS, vol.195, no. 912, AMS, Providence, RI, 2008.
- [15] I. Chueshov and I. Lasiecka, *Von Karman Evolution Equations*, Springer, New York, 2010.
- [16] D. Coutand, S. Shkoller, Motion of an elastic solid inside an incompressible viscous fluid, *Arch. Ration. Mech. Anal.* **176** (2005), 25–102.
- [17] G. Galdi, C. Simader, H. Sohr, A class of solutions to stationary Stokes and Navier-Stokes equations with boundary data in $W^{-1/q,q}$, *Math. Annalen* **331** (2005), 41–74.
- [18] Q. Du, M.D. Gunzburger, L.S. Hou, J. Lee, Analysis of a linear fluid–structure interaction problem, *Discrete Contin. Dyn. Syst.* **9** (2003), 633–650.
- [19] M. Grobbelaar-Van Dalsen, On a fluid-structure model in which the dynamics of the structure involves the shear stress due to the fluid, *J. Math. Fluid Mech.* **10** (2008), 388–401.
- [20] M. Grobbelaar-Van Dalsen, A new approach to the stabilization of a fluid-structure interaction model, *Applicable Analysis* **88** (2009), 1053–1065.
- [21] M. Grobbelaar-Van Dalsen, Strong stability for a fluid-structure model, *Math. Methods Appl. Sci.* **32** (2009), 1452–1466.
- [22] N. Kopachevskii, Yu. Pashkova, Small oscillations of a viscous fluid in a vessel bounded by an elastic membrane, *Russian J. Math. Phys.* **5** (1998), no.4, 459–472.

- [23] O. Ladyzhenskaya, *Mathematical Theory of Viscous Incompressible Flow*, GIFML, Moscow, 1961 (1st Russian edition); Nauka, Moscow, 1970 (2nd Russian edition); Gordon and Breach, New York, 1963 and 1969 (English translations of the 1st Russian edition).
- [24] J. Lagnese, *Boundary Stabilization of Thin Plates*, SIAM, Philadelphia, 1989.
- [25] J. Lagnese, Modeling and stabilization of nonlinear plates, *Int. Ser. Num. Math.*, **100** (1991), 247–264.
- [26] J. Lagnese and J.L.Lions, *Modeling, Analysis and Control of Thin Plates*, Masson, Paris, 1988.
- [27] J.-L. Lions, E. Magenes, *Problèmes aux limites non homogènes et applications*, Vol. 1, Dunod, Paris, 1968.
- [28] J.-L. Lions, *Quelques methodes de resolution des problèmes aux limites non lineaire*, Dunod, Paris, 1969.
- [29] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1986.
- [30] J. Simon, *Compact sets in the space $L^p(0, T; B)$* , Annali di Matematica Pura ed Applicata, Ser.4 **148** (1987), 65–96.
- [31] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer, New York, 1988.
- [32] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, Reprint of the 1984 edition, AMS Chelsea Publishing, Providence, RI, 2001.
- [33] H. Triebel, *Interpolation Theory, Functional Spaces and Differential Operators*, North Holland, Amsterdam, 1978.